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: Matrices :

Definition of Matrix :

The rectangural array (arrangement) of mn numbers enclosed within a bracket with m horizontal rows and n vertical colums is called a matrix of order m×n.

For ex:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

(or)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Note:

① In a matrix nos of rows needs not to be equal matrix of order (m×n) to no of colomns.

② Above matrix can be abbreviated as

$A = [a_{ij}]_{m\times n}$ (or) $(a_{ij})_{m\times n}$

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③ Unlike determinant matrix has no value and subjected to certain rules.

④ If all the elements of matrix are real then matrix is called real matrix.

Q. Construct a matrix of order 3x2 with

$$a_{ij} = \frac{|i-2j|}{2}.$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{31} \end{bmatrix}_{3\times2}$$

$a_{11} = \left|\frac{1-2}{2}\right| = 1/2$

$a_{12} = \left|\frac{1-4}{2}\right| = 3/2$

$a_{21} = \left|\frac{2-2}{2}\right| = 0$

$a_{22} = \left|\frac{2-4}{2}\right| = 1$

$a_{31} = \left|\frac{3-2}{2}\right| = 1/2$

$a_{32} = \left|\frac{3-4}{2}\right| = 1/2$

$$A = \begin{bmatrix} 1/2 & 3/2 \\ 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

Types of Matrices:

1) Row Matrices

A matrix having only one row is called row matrix.

Ex: $A = [a_{11} \ a_{12} \ a_{13} \ \cdots \ a_{1n}]$

$A = [1 \ 2 \ 3 \ 4 \ 5]_{1\times5}$

2) Column Matrix:

A matrix having only one column is called column Matrix.

$$A = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 10 \end{bmatrix}_{4\times1} = \text{Column Matrix.}$$

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3) Null Matrix (or Zero Matrix)

A Matrix having all its elements equal to zero is called null Matrix.

A null Matrix of order $m \times n$ is written as $O_{m \times n}$

Ex: $O_{3\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $O_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4) Horizontal Matrix: $r > c$

If no of rows < no of columns then matrix will be Horizontal Matrix.

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}_{2\times4}$.

5) Vertical Matrix: $c > r$.

If no of rows > no of columns then matrix will be vertical Matrix

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times2}$.

6) Square Matrix:

If no. of rows = no of columns then matrix is called square matrix

Square matrix of n means having n rows and n columns.

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H/W
Ex: 3-B AOD

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Note:

① Elements $a_{11}, a_{22}, a_{33} \cdots a_{nn}$ are called diagonal (or principle diagonal) elements.

② Trace of a square matrix A = Sum of diagonal elements

$$tr(A) = a_{11} + a_{22} + a_{33} \cdots a_{nn} = \sum_{i=1}^{n} a_{ii}$$

③ Elements a_{ij} and a_{ji} are called conjucate elements of each other.

Types of Square Matrices:

1) Upper triangular Matrix:

A square matrix $A = [a_{ij}]_{m \times n}$ is said to be upper triangular if $a_{ij} = 0 \;\; \forall \; i > j$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

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2) Lower triangular Matrix:

A square matrix $A = [a_{ij}]$ is said to be lower triangular if $a_{ij} = 0 \ \forall \ i<j$.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3) Diagonal Matrix:

A square matrix is said to be diagonal if it is lower as well as upper triangular (or) If $a_{ij} = 0 \ \forall \ i \neq j$ ~~th~~

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \qquad A = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$\parallel$$

$$\text{dia}(d_1, d_2, d_3 \cdots d_n).$$

Types of diagonal Matrix

1) Scalar Matrix:

If all the diagonal elements of diagonal matrix are equal to k $(k \in R)$ then matrix is called scalar matrix.

$$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = \text{Scalar Matrix.}$$

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If $k=1$ then it is called identity matrix and denoted by I

I_n = identity matrix of order n.

$$I_2 = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{Unit Matrix.}$$

Note:

① Corresponding to every square matrix A there exisist a determinant having same corresponding terms denoted as det (A) or |A|

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \text{then} \quad \det(A) = |A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\Rightarrow$ If $\det(A) = 0$ then matrix A is called singular Matrix.

$\Rightarrow$ If $\det(A) \neq 0$ then matrix A is called as non signgolar matrix.

Q. Find minimum number of zeros in a triangular matrix (upper or lower) of order n

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & & a_{mn} \end{bmatrix} = 1+2+\cdots+(n-1)$$

$$= \frac{n(n-1)}{2}$$

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Q. Find the minimum and maximum number of zeros in a diagonal matrix.

minimum No $= 2 \times \frac{n(n-1)}{2} = n^2 - n$

Maximum $= n^2 - 1$

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

Algebra of Matrices:

1) Equality of two matrices:

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

(i) They are of same order

(ii) Each corresponding elements should be same.

$$a_{ij} = b_{ij} \quad \forall \ i \ \& \ j$$

Ex:

$$A = \begin{bmatrix} a & b & 3 \\ -7 & 5 & c \end{bmatrix} = \begin{bmatrix} -8 & 5 & d \\ e & \delta & 4 \end{bmatrix}$$

$a = -8, \ b = 5, \ c = 4, \ d = 3, \ e = -8, \ \delta = 5$

2) Addition or Subtraction of two matrices:

Let A & B be two matrices, $A \pm B$ will exisist iff order of A = Order of B.

$A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$

$$A \pm B = [a_{ij} \pm b_{ij}]_{m \times n}.$$

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For ex:

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A+B = \begin{bmatrix} 2+a & 3+b \\ 5+c & 7+d \end{bmatrix}$$

Note:

① $A+B = B+A$

② $A+(B+C) = (A+B)+C$

③ If $A+B=0$ then B is called additive inverse of A.

④ $tr(A+B) = tr(A) + tr(B)$

⑤ Multiplication of Scalar in a Matrix

$\Rightarrow$ Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$k \in R \quad kA = \begin{bmatrix} ka_1 & ka_2 & ka_3 \\ kb_1 & kb_2 & kb_3 \\ kc_1 & kc_2 & kc_3 \end{bmatrix}$

⑥ Let A be a square matrix of order n then $|kA| = k^n |A| \quad (k \in R)$.

⑦ $tr(kA) = k\, tr(A)$

⑧ If $A+B = A+C$ then $B=C$

Q. Solve the equation:

$[x \quad 2y \quad 3z] - 2[y \quad z \quad -x] + 3[-z \quad x \quad y] = [-12, 1, 17]$

$[x-2y-3z \quad 2y-2z+3x \quad 3z+2x+3y] = [-12, 1, 17]$

$$x - 2y - 3z = 12 \longrightarrow (1)$$
$$3x + 2y - 2z = 1 \longrightarrow (2)$$
$$2x + 3y + 3z = 17 \longrightarrow (3)$$

$x = 1, \; y = 2, \; z = 3$

Q. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then find $2A + 3B - 5I$

$$5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad 2A = \begin{bmatrix} 2 & 4 & 0 \\ -4 & 2 & 6 \\ 0 & -2 & 4 \end{bmatrix}$$

$$3B = \begin{bmatrix} 3 & -6 & 6 \\ 6 & 9 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$

$$2A + 3B + 5I = \begin{bmatrix} 0 & -2 & 6 \\ 2 & 6 & 9 \\ 0 & 1 & -1 \end{bmatrix}$$

Q. Find the matrices x and y if

$$2x - y = \begin{pmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{pmatrix} \longrightarrow (1) \quad \& \quad x + 2y = \begin{pmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{pmatrix} \longrightarrow (2)$$

Eqn (1) × 2

$$5x = \begin{pmatrix} 15 & -10 & 5 \\ -10 & 5 & -5 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$$

(1) − (2) × 2

$$-5y = \begin{pmatrix} 0 & -10 & -10 \\ 0 & 0 & 15 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & -3 \end{pmatrix}$$

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Q. A matrices has 12 elements. Find the no of possible orders it can have.

Q. Let A be a square matrix of order n

l = Maxm number of distinct entries of A is triangular

m = " " " " " A in diagonal

P = Minim " " zeros if A is triangular.

(i) Matrix has 12 elements.

$\downarrow$

$m \times n$

$m \times n = 12 = 1, 2, 3, 4, 6, 12$

$= 1 \times 12,\ 2 \times 6,\ 3 \times 4$

$12 \times 1,\ 6 \times 2,\ 4 \times 3$

$\therefore$ No of different orders $= 6$.

(ii)
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn} \end{bmatrix} = n + (n-1) + (n-2) + \cdots + 2 + 1$$

$$= \frac{n(n+1)}{2}$$

$$l = (n + (n-1) + (n-2) + \cdots + 2 + 1) + 1$$

$$= \frac{n(n+1)}{2} + 1$$

$$m = n + 1$$

$$p = \frac{n(n-1)}{2}$$

$$\Rightarrow \frac{n(n+1)}{2} + 1 + 5 = \frac{n(n-1)}{2} + 2 \cdot (n+1)$$

$$\Rightarrow \cancel{n^2} + n + 12 = \cancel{n^2} - n + 4n + 4$$

$$\Rightarrow 2n = 8 \Rightarrow n = 4.$$

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Multiplication of two Matrices (Row to Column)

Let A and B be two non zero matrices then product AB will exist iff

no of columns in A = no of rows in B.

For. ex,

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}_{2\times 3} \text{ and } B = \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{bmatrix}_{3\times 2}$$

$$\therefore AB = \begin{bmatrix} a_1 l_1 + b_1 l_2 + c_1 l_3 & a_1 m_1 + b_1 m_2 + c_1 m_3 \\ a_2 l_1 + b_2 l_2 + c_2 l_3 & a_2 m_1 + b_2 m_2 + c_2 m_3 \end{bmatrix}_{2\times 2}$$

Note:

① If order of $A = m\times n$ and order of $B = n \times p$ then $A\times B$ will exist and its order $= m\times p$

$A_{m\times n} \; B_{n\times p}$

② If AB exists then BA need not to exist (It may or maynot exist).

③ In general $AB \neq BA$ (even if both exists and have same order).

④ If A and B are two matrices such that $AB = BA$, then A and B commutative matrices.

⑤ If $AB = -BA$ then A and B are called anti commutative matrices.

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$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad A \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$AB \neq BA$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = I \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = IA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(6) Identity matrix is commutative with every square matrix of same order.

$$\boxed{AI = IA = A}$$

(7) If $AB = 0 \not\Rightarrow$ either $A = 0$ (or) $B = 0$

For ex:

$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$ & $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

(8) If any one of A and B is null matrix and product AB exists then $AB = 0$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

(9) $|AB| = |A||B|$ (Where A and B are square matrices of same order)

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(10). If $AB = 0 \Rightarrow |AB| = 0 \Rightarrow |A||B| = 0$

$\Rightarrow$ either $|A| = 0$ (or) $|B| = 0$.

(11) If $AB = AC \not\Rightarrow B = C$ if $A \neq 0$.

$\Downarrow$

$AB - AC = 0$

$A(B-C) = 0$

$A \neq 0 \quad B-C \neq 0$

$\Rightarrow B \neq C$

(12) $A(B \pm C) = AB \pm AC$

(13) $A(BC) = (AB)C$

But $ABC \neq ACB$

(14) If A is null matrix then $|A| = 0$

But if $|A| = 0$ then A need not to be null matrix.

(15) A^2 means AA will exist if A is square matrix.

$A_{m \times n} \quad A_{m \times n}$

$A^n = \underbrace{A \cdot A \cdot A \cdots A}_{n \text{ times}}$.

(16) $(A+B)^2 \neq A^2 + B^2 + 2AB$

$(A+B)^2 = (A+B)(A+B)$

$= A \cdot A + AB + BA + B \cdot B$

$= A^2 + AB + BA + B^2$

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17) If A and B are commutative matrices.

$(A+B)^2 = A^2 + B^2 + 2AB$.

$(A+B)(A-B) = A^2 - B^2$

$(A+B)^3 = A^3 + B^3 + 3A^2B + 3AB^2$.

$(A+B)^n = {}^nC_0 A^n + {}^nC_1 A^{n-1}B + {}^nC_2 A^{n-2}B^2$

$\ldots\ldots + {}^nC_{n-1} AB^{n-1} + {}^nC_n B^n$.

18) Let $A = \text{dia}(a_1, a_2, a_3 \ldots a_n)$

$B = \text{dia}(b_1, b_2, b_3 \ldots b_n)$

$AB = \text{dia}(a_1b_1, a_2b_2, a_3b_3 \ldots a_nb_n)$.

& $A^2 = \text{dia}(a_1^2, a_2^2, a_3^2 \ldots a_n^2)$

$A^n = \text{dia}(a_1^n, a_2^n, a_3^n \ldots a_n^n)$

19) $I^2 = I$ $\quad I = \text{dia}(1, 1, 1)$

$= I^3 = \ldots I^n$ $\quad I^2 = \text{dia}(1^2, 1^2, 1^2)$

$= I$

20) A^0 means identity matrix of same order as of A

$\Rightarrow A^0 = I_n$

Matrix Polynomial:

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1} + a_n$, be a polynomial (where $a_0 \neq 0$) and A be any square matrix

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then $f(A) = a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I$
is called a matrix polynomial.
and if $f(A) = 0$ then A is called a Zero or root of $f(A)$.

Q. Matrix A has m rows and $(n+5)$ columns
Matrix B has m rows and $(11-n)$ columns
If both AB and BA exists find m & n.

$AB = n+5 = m$

$BA = 11-n = m$

$n + 5 = 11 - n \Rightarrow 2n = 6$

$\Rightarrow n = 3$.

$\therefore m = 8$.

Q. Find x such that matrix product

$$\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$$ equals an identity Matrix.

$$\begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x & 5x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$5x = 1$ & $10x - 2 = 0$

$x = 1/5$

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Q. Given that order of $A = m\times n$ order of $B = n\times p$ and find condition $m\ n\ p\ r\ q$ for exisistance of

a) ABC b) ACB c) $A(B+C)$.

a) $A_{m\times n}$ $B_{n\times p}$ $C_{r\times q}$

$= (ABC)_{m\times q}$

$\boxed{\therefore p = r}$

b) $A_{m\times n}$ $C_{r\times q}$ $B_{n\times p}$

Condition: $\Rightarrow n = r = q$

$(ACB)_{m\times p}$

c) $B_{n\times p} + C_{r\times q}$ will exisist

Condition
$n = r$
$p = q$

$A_{m\times n}(B+C)_{n\times p}$

Q. Find all matrices which commute with matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}$$

$$\therefore AB = BA$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

$$a + c = a \Rightarrow c = 0$$
$$b + d = a + b \Rightarrow a = d$$
$$d = c + d \Rightarrow c = 0$$

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Q. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ & $B = \begin{pmatrix} a & +1 \\ b & -1 \end{pmatrix}$

If $(A+B)^2 = A^2 + B^2$ find a and b.

$$\because (A+B)^2 = A^2 + B^2$$

$$A^2 + B^2 + AB + BA = A^2 + B^2$$

$$AB = -BA$$

$$\Rightarrow \begin{pmatrix} a-b & 2 \\ 2a-b & 3 \end{pmatrix} = -\begin{pmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{pmatrix}$$

$$2 = a+1 \Rightarrow a = 1$$
$$3 = b-1 \Rightarrow b = 4.$$

Q. Find an upper triangular matrix A such that $A^3 = \begin{pmatrix} 8 & -57 \\ 0 & 27 \end{pmatrix}$

Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

$$A^2 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & ab+bc \\ 0 & c^2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} a^2 & ab+bc \\ a & c^2 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^3 & a^2b + abc + bc^2 \\ 0 & c^3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -57 \\ 0 & 27 \end{pmatrix}$$

$$\therefore a^3 = 8 \Rightarrow a = 2$$
$$c^3 = 27 \Rightarrow c = 3$$

$$a^2b + abc + bc^2 = -57$$
$$\Rightarrow 19b = -57 \Rightarrow b = -3.$$

$$A = \begin{pmatrix} 2, & -3 \\ 0 & 3 \end{pmatrix}$$

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Q. Let

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{pmatrix}$$

$$AC = \begin{pmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{pmatrix}$$

$$AB = AC.$$

Q. Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$ and $f(x) = x^2 - 4x + 7$ then prove that $f(A) = 0$. Use this result to find A^3.

$$f(A) = A^2 - 4A + 7I$$

$$A^2 = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 12 \\ -4 & 1 \end{pmatrix}$$

$$4A = \begin{pmatrix} 8 & 12 \\ -4 & 8 \end{pmatrix}$$

$$7I = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$$

$$A^2 - 4A + 7I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$A^2 = 4A - 7I$$

$$A^3 = A^2 \cdot A = (4A - 7I)\,A$$

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$$A^3 = 4A^2 - 7A$$
$$= 4(4A - 7I) - 7A.$$
$$= 9A - 28I$$
$$= \begin{pmatrix} 18 & 27 \\ -9 & 18 \end{pmatrix} - \begin{pmatrix} 28 & 0 \\ 0 & 28 \end{pmatrix}$$
$$= \begin{pmatrix} -10 & 27 \\ -9 & -10 \end{pmatrix}$$

Characteristic equation (or Cayley Hamiltons Theory).

Every square matrix A satisifies a characteristic equation ie. $\boxed{|A - xI| = 0}$ $x \in R$

$$A - xI = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$
$$= \begin{pmatrix} 2-x & 3 \\ -1 & 2-x \end{pmatrix}$$
$$(A - xI) = \begin{vmatrix} 2-x & 3 \\ -1 & 2-x \end{vmatrix} = 0$$
$$\Rightarrow (2-x)^2 + 3 = 0$$
$$\Rightarrow x^2 - 4x + 7 = 0$$

If A is square matrix of order 2 then its characteristic equation will be
$$x^2 - (tr\,A)x + |A| = 0.$$

Q. Find the characteristic equation of Matrix $A = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

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$|A - xI| = 0$

$$xI = \begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$$

$$|A - xI| = \begin{vmatrix} 2-x & -1 & 1 \\ -1 & 2-x & -1 \\ 1 & -1 & 2-x \end{vmatrix} = 0$$

$\Rightarrow (2-x)((2-x)^2-1) + 1((x+2)+1) + 1(1-2+x) = 0$

$\Rightarrow (2-x)(x^2-4x+3) + 2(x-1) = 0$

$\Rightarrow -x^3 + 6x^2 - 9x + 4 = 0$

$\Rightarrow x^3 - 6x^2 + 9x - 4 = 0$

Q. For $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ find a & b such that

$A^2 + aA + bI = 0$

$x^2 + ax + b = 0$

$a = -tr(A) = -4$

$b = |A| = 1.$

Some Special type of Square Matrices:

1) Indempodent Matrix:

Let A be a square matrix such that $A^2 = A$ then matrix is called idempotent matrix.

Note:

If A is idempotent then

$A^2 = A$

$A^3 = A^2 \cdot A = A \cdot A = A^2 = A$

$\therefore A = A^2 = A^3 = A^4 \ldots \ldots$

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2) Nilpotent Matrix:

Let A be square matrix such that $A^k = 0$ where A^{k-1} is $\neq 0$ then A is called nilpotent of order or index k.

If $A^k = 0$

then $A^{k+1} = A^{k+2} = \cdots = 0$

$$A^4 = A^3 \cdot A = 0 \cdot A = 0 .$$

3) Periodic Matrix:

If $A^{k+1} = A$ then A is called periodic matrix with period k where k is least possible positive integral value.

If A ∋ $A^2 \neq A$, $A^3 = A$, $\Rightarrow$ Period 2.

Note:

Every indempotent is periodic with period 1.

4) Involutary Matrix:

If $A^2 = I$ then A is involutary Matrix

Note:

If A is involutary

then $A^3 = A^2 \cdot A = I \cdot A = A$

$\therefore$ A is periodic with period 2.

$A^4 = (A^2)^2 = I^2 = I$, $A^5 = A$.

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Q. Matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & 3 \end{bmatrix}$ is

(a) Indempotent (b) Nilpotent (c) periodic (d) Involuntry.

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A.$$

Q. $A = \begin{bmatrix} ab & b^2 \\ a^2 & -ab \end{bmatrix}$

$$A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$\therefore$ Nilpotent with index 2.

Q. $A = \begin{bmatrix} a & -a^2 \\ 1 & -a \end{bmatrix}$

$$A^2 = \begin{bmatrix} a & -a^2 \\ 1 & -a^2 \end{bmatrix} \begin{bmatrix} a & -a^2 \\ 1 & -a^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Nilpotent.

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H/W: 61, 62, 63 DPP's.

Q. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = $ ~~Ando~~.

Q. If A is involuntary matrix then prove that $(I-A)(I+A) = 0$.

$\because A^2 = I$.

LHS $= (I-A)(I+A)$

$= I^2 - A^2$

$= I^2 - I^2 = 0$ ($\because$ A is involuntary).

Q. If A is periodic with period 3 and $A^6 + B = I$ then $AB =$

(a) I (b) 0 (c) A^2 (d) $A + I$.

$A^6 + B = I$

$A \cdot A^6 + AB = A \cdot I$.

$A + AB = A \Rightarrow AB = 0$

Transpose of a matrix:

A matrix obtained by changing rows into column and column into rows is called transpose of a matrix.

Let A be $= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

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Transpose of $A = A^T = A' = \bar{A} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}_{3\times 2}$

Note:

① If $A = [a_{ij}]_{m\times n}$.

then $A^T = A' = \cancel{[a_{ij}]_{m\times n}}\ [a_{ji}]_{n\times m}$

② $(A^T)^T = A$.

③ $(A \pm B)^T = A^T \pm B^T$

④ $(A\cdot B)^T = B^T \cdot A^T$.

⑤ $(A_{m\times n} B_{n\times p})^T = B^T_{p\times n} A^T_{n\times m}$

In general $(A_1, A_2, A_3 \cdots A_n)^T = A_n^T \cdot A_{n-1}^T \cdots A_2^T \cdot A_1^T$

⑤ $(A^n)^T = (A^T)^n$

⑥ If A is square Matrix.

$|A| = |A^T|$

Symmentric and Skew Symmentric Matrices

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be symmentric

if $a_{ij} = a_{ji}$ $\forall$ a_{ij} & $a_{ji} \in A$.

If $A = A^T$ then A is called symmentric

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$a_{11} = a_{11}$

$a_{12} = a_{21}$

$a_{13} = a_{31}$

$a_{23} = a_{32}$

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$$\therefore A^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = A$$

If $a_{ij} = -a_{ji}$ $\forall i \& j$ then square matrix A is called skew symmentric

(or)

If $A^T = -A$ then A is called Skew Matrix.

$a_{11} = -a_{11} \Rightarrow a_{11} = 0$ $\quad a_{ij} = -a_{ij}$

$a_{22} = -a_{22} \Rightarrow a_{22} = 0$ $\quad a_{ii} = 0$

$a_{12} = -a_{21}$ $\quad a_{13} = -a_{31}$

For ex: $A = \begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & -2 \\ -7 & 0 & 2 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 0 & -5 & -7 \\ 5 & 0 & 2 \\ 7 & -2 & 0 \end{bmatrix} = -A$

Note:

① In a skew symmentric Matrix all diagonal elements are equal to zero.

② In a skew symmentric, conjucate elements are additive inverse of each other.

③ If A is a square skew symmentric of odd order then $|A| = 0$

$\therefore A^T = -A$.

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Proof:

$\therefore A^T = A$

$\Rightarrow |A^T| = |-A|$

$|A| = (-1)^{2n-1}|A|$ (where order of A is 2n-1)

$\Rightarrow |A| = -|A| \Rightarrow 2|A| = 0$

$|A| = 0$

④ Maximum number of distinct elements in a symmentric matrix of order n $= \frac{n(n+1)}{2}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \times & a_{22} & a_{23} & \cdots & a_{2n} \\ \times & \times & a_{33} & \cdots & a_{3n} \\ - & - & - & \cdots & a_{mn} \end{bmatrix}$$

$\Rightarrow n + (n-1) \cdots + 1$

$= \frac{n(n+1)}{2}$

⑤ If a square metrix A which is symmentric as well as skew symmentric then A must be null Matrix.

⑥ Every square matrix a can be written as sum of symmentric and skew symmentric Matrices.

$$A = \underbrace{\frac{(A+A^T)}{2}}_{\text{Symmentric}} + \underbrace{\frac{(A-A^T)}{2}}_{\text{Skew symmentric}}$$

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$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$\Rightarrow$ Symmentric

$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$ skew

7) If A is any square matrix then

(i) $A + A^T$ will be symmentric

(ii) $A - A^T$ " " skew symmentric

(iii) AA^T (or) A^TA will be symmentric

$(AA^T)^T = (A^T)^T \cdot (A)^T = AA^T$ symmentric.

8) If A is symmentric Matrix then A^n will be symmentric

$(A^n)^T = (A^T)^n = A^n \Rightarrow$ Symmentric $(n \in N)$

9) If A is a skew matrix then

A^n will be $= \begin{cases} \text{symmentric} & n = \text{even} \\ \text{Skew symmentric} & n = \text{odd} \end{cases}$

Proof: $\because A^T = -A$

$(A^n)^T = (A^T)^n = (-A)^n = \begin{cases} A^n, & n = \text{even} \\ -A^n, & n = \text{odd}. \end{cases}$

Q. If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$ and $B = [1, 3\ -6]$

Verify $(AB)^T = B^T \cdot A^T$.

$AB = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}_{3 \times 3}$

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$$AB^T = \begin{bmatrix} -2 & 4 & \cancel{-30}\,5 \\ -6 & 12 & 15 \\ -12 & -24 & -30 \end{bmatrix}$$

$$B^T . A^T = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \begin{bmatrix} -2 & 4 & 5 \end{bmatrix}_{1\times3}$$

$$= \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$$

Hence Verified.

Q. Prove that the matrix $B^T AB$ is symmentric or skew symmentric according as A is symmentric or skew symmentric.

$$(B^T AB)^T = B^T . A^T (B^T)^T$$

$$= B^T . A^T . B$$

$$= \begin{cases} B^T AB & \text{if A is symm} \\ -B^T AB & \text{if A is skew sym} \end{cases}$$

Q. Let A & B be symmentric matrices of same order then prove that.

(i) A+B is symmentric

(ii) AB+BA is symmentric

(iii) AB−BA is skew symmentric.

(i) $A^T = A$, $B^T = B$.

$(A+B)^T = A^T + B^T = A+B$

$\therefore$ Symm .

(ii) $(AB+BA)^T = (AB)^T + (BA)^T$.

$= B^T A^T + A^T B^T$

$= BA + AB = AB + BA$ symm .

(iii) ~~(A-B)~~

$(AB - BA)^T = (AB)^T - (BA)^T$

$= B^T A^T - A^T B^T$

$= BA - AB = -(AB - BA)$ Skew Symm .

Q. If $A = \begin{bmatrix} +1 & 2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & y-x & -1 \\ x & 1 & 2 \end{bmatrix}$

such that AB is symm matrix then find x and y.

$$AB = \begin{bmatrix} 1-2x & y-x-2 & -5 \\ 3+x & 3y-3x+1 & -7 \\ -1+2x & x-y+2 & 5 \end{bmatrix}$$

$2x - 1 = -5 \Rightarrow x = -2$

$x - y + 2 = -1 \Rightarrow y = 1$

Q. Express $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ as sum of two matrices one symm and other skew symm

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$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$A^T = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}}_{\text{symm}} + \underbrace{\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}}_{\text{skew}}$$

Q. If $A = \begin{bmatrix} 5 & 2 & x \\ y & z & -3 \\ 4 & 6 & 7 \end{bmatrix}$ is sym find x, y, z, t.

$y = 2$, $x = 4$, $t = -3$, $z \in R$

Orthogonal Matrix :

Any Matrix A is said to be orthogonal if $AA^T = A^T \cdot A = I$.

Note :

① If A is a square orthogonal matrix then $|A| = \pm 1$

Proof:

$$A \cdot A^T = I$$

$$\Rightarrow |AA^T| = |I|$$

$$|A|\,|A^T| = 1 \qquad (\because |AB| = |A|\,|B|)$$

$$|A|^2 = 1$$

$$|A| = \pm 1$$

Ex: $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

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Q. If A and B are orthogonal matrices of same order then prove that AB is orthogonal.

$$(AB)\cdot(AB)^T = A(B\cdot B^T)A^T$$
$$= A\cdot I\cdot A^T = AA^T = I$$

$\therefore$ AB is orthogonal.

Q. If M is a square 3x3 matrix where $M^TM = I$ then find $|M-I|$.

$$|M-I| = |M - M^TM|$$
$$= |(I-M^T)M|$$
$$= |I-M^T|\,|M|$$
$$= |(I-M)^T| \cdot 1 = |I-M|$$
$$= |-(M-I)|$$

$$|M-I| = (-1)^3|M-I|$$
$$= -|M-I|$$

$$\Rightarrow |M-I| = 0.$$

Adjoint of a Square Matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Matrix of cofactors of elements of corresponding determinatns (A) $= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

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adjoint (A) = adjoint of square matrix A

= It is transpose of matrix of cofactors of elements of corresponding determinant.

$$= \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Note:

① Let $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ $c_{11} = s$ $c_{12} = -r$

$c_{21} = -q$ $c_{22} = p$

matrix of cofactors = $\begin{bmatrix} s & -r \\ -q & p \end{bmatrix}$

adj (A) = $\begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$

Adjoint of a square matrix of order 2 can be directly calculated by interchanging diagonal elements of and by changing the sign of other elements.

② Adjoint of a diagonal, scalar, triangular will be diagonal, scalar, triangular respectively

*** ③ A · (adj A) = |A| · I_n (where I_n is identity matrix of order n)

(i.e order of A).

Proof:

$$LHS = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & & \vdots \\ c_{1n} & c_{2n} & & c_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

④ $|adj(A)| = |A|^{n-1}$ (n = order of A)

Proof:

$$A\cdot(adj A) = |A| I_n .$$

$$|A\cdot(adj A)| = ||A| I_n| .$$

$$|A|\,|adj A| = |A|^n |I_n| .$$

$$|adj A| = |A|^{n-1}$$

⑤: $adj(adj A) = |A|^{n-2} . A$

Proof:

$$A\ adj(A) = |A|\cdot I_n .$$

$$\Rightarrow (adj A) \cdot adj(adj(A)) = |adj(A)| \cdot I_n$$

$$\Rightarrow A (adj A) \cdot adj(adj(A)) = |A|^{n-1} \cdot A \cdot I_n$$

$$\Rightarrow |A| I_n\ adj(adj A) = |A|^{n-1} A$$

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$$\Rightarrow \boxed{\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} \cdot A}$$

⑥ $|\operatorname{adj}(\operatorname{adj} A)| = |A|^{(n-1)^2}$.

$\therefore |\operatorname{adj}(A)| = |A|^{n-1}$

$\therefore |\operatorname{adj}(\operatorname{adj} A)| = |\operatorname{adj}(A)|^{n-1}$

$= \left(|A|^{(n-1)}\right)^{n-1}$

$= |A|^{(n-1)^2}$

⑦ $|(\operatorname{adj}(\operatorname{adj}(\operatorname{adj} A)))| = |A|^{(n-1)^3}$.

⑧ $\operatorname{adj}(AB) = \operatorname{adj}(B) \cdot \operatorname{adj}(A)$.

⑨ $\operatorname{adj}(A^T) = (\operatorname{adj}(A))^T$

⑩ $\operatorname{adj}(KA) = k^{n-1} \operatorname{adj}(A)$. ($n$ = order of A). ($k \in R$).

⑪ If a symmentric matrix then adj A is also a symmentric Matrix.

Inverse (or reciprocal) of a square Matrix:

Let A be any square non singular matrix then there exist a unique square matrix B of same order such that

$$AB = BA = I_n \quad (n = \text{order of } A)$$

then B is called inverse of A

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It is denoted as A^{-1} $\boxed{B = A^{-1}}$ $= 1/A$.

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

Proof:

$$AB = I_n .$$

$$\text{adj}(A) \cdot A \cdot B = I_n \text{ adj}(A) .$$

$$|A| I_n \cdot B = \text{adj}(A)$$

$$B = \frac{1}{|A|} \text{adj}(A) = A^{-1} .$$

Note:

① The necessary and sufficient condition for inverse to exist is $|A| \neq 0$.

② Inverse of a square invertible matrix A will be unique.

③ Let $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ then $A^{-1} = \frac{1}{|A|} \text{adj} A = \frac{1}{|A|} \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$

④ $(A^{-1})^{-1} = A$

⑤ $(AB)^{-1} = B^{-1} A^{-1}$

In general
$(A_1 A_2 A_3 \cdots A_{n-1} A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \cdots A_2^{-1} A_1^{-1}$

⑥ $(A^n)^{-1} = (A^{-1})^n$

⑦ $|A^{-1}| = |A|^{-1} = \frac{1}{|A|}$

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Proof:

$$\because\ A^{-1} = \frac{1}{|A|}\,\text{adj}(A).$$

$$|A^{-1}| = \left|\frac{1}{|A|}\,\text{adj}(A)\right|$$

$$= \frac{1}{|A|^n}\,|\text{adj}(A)| = \frac{|A|^{n-1}}{A^n} = \frac{1}{|A|}$$

(8) $(A^T)^{-1} = (A^{-1})^T$

(9) If $A = \text{dia}(d_1, d_2, d_3 \cdots d_n)$

then $A^{-1} = \text{dia}(d_1^{-1}, d_2^{-1}, d_3^{-1}, \cdots d_n^{-1})$

(10) If A is idempotent invertible matrix

then $A^2 = A$

$$A^{-1}\cdot A\cdot A = A^{-1}\cdot A$$

$$I\cdot A = I \Rightarrow A = I.$$

$$A^{-1} = A = I.$$

(11) If A is involutary invertible matrix then

$A^{-1} = A$

Proof: ~~If~~ A ~~is~~ orthogonal ~~square invertible~~

$$A^2 = I \Rightarrow A^{-1}\cdot A\cdot A = A^{-1} I$$

$$\Rightarrow A = A^{-1}$$

(12) If A is orthogonal square invertible matrix

then $A^{-1} = A^T$

$$\therefore\ AA^T = I$$

$$\Rightarrow A^{-1}AA^T = A^{-1}\cdot I \Rightarrow A^T = A^{-1}.$$

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Q. Find adjoint of square matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 4 \\ 2 & 6 & 7 \end{bmatrix}$$

$C_{11} = -24$ $\quad C_{12} = -(35-8) = -27$ $\quad C_{13} = 30$

$C_{21} = -(14-18) = 4$ $\quad C_{22} = 7-6 = 1$ $\quad C_{23} = -(6-4) = -2$

$C_{31} = 8$ $\quad C_{32} = -(4-15) = 11$ $\quad C_{33} = (0-10) = -10$

$$\begin{bmatrix} -24 & -27 & 30 \\ 4 & 1 & -2 \\ 8 & 11 & -10 \end{bmatrix} \qquad adj(A) = \begin{bmatrix} -24 & 4 & 8 \\ -27 & 1 & 11 \\ 30 & -2 & -10 \end{bmatrix}$$

Board:

Finding inverse Using elementry transformation

Ex: Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ } format.

$\therefore A = AI$

$$\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} = A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = A\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} = A\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

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$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 4 \\ 0 & -7 \end{bmatrix} = A \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$$

$R_2 \to \frac{1}{-7} R_2$

$$\Rightarrow \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} -1 & 1 \\ -3/7 & 2/7 \end{bmatrix}$$

$$= A \begin{bmatrix} 5/7 & -1/7 \\ -3/7 & 2/7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 5/7 & -1/7 \\ -3/7 & 2/7 \end{bmatrix}$$

Q. Find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ using elementry operations.

$A = IA$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A.$$

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$R_3 \to R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A.$$

$R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} \times A.$$

$R_3 \to \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A.$$

$R_1 \to R_1 + R_3$ $\quad R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}}_{A^{-1}} A$$

Q. If A B C are $n \times n$ matrices such that $|A| = 2$, $|B| = 3$, $|C| = 5$ then find $|A^2 B C^{-1}|$

$$|A^2 B C^{-1}| = |A^2|\,|B|\,|C^{-1}|$$

$= |A^2|\ |B|\ |C^{-1}|$

$= |A^2| \cdot |B| \cdot \frac{1}{|C|} = 4 \cdot 3 \cdot 1/5$

$= 12/5$

Q. If A and B are square matrices of order 3 such that $|A| = -2$, $|B| = 1$, then find $|A^{-1} \cdot \text{adj}(B^{-1}) \text{ adj}(2A^{-1})|$.

$= |A^{-1}|\ |\text{adj}(B^{-1})|\ |\text{adj}(2A^{-1})|$

$= \frac{1}{|A|} \times |B^{-1}|^{3-1}\ |2A^{-1}|^{3-1}$.

$= \left(\frac{1}{|A|} \times \left(\frac{1}{|B|}\right)^2\right) \times \left(2^3 |A^{-1}|\right)^2$

$= \frac{1}{-2} \times 1 \times \left(8 \times \frac{1}{-2}\right)^2$

$= \frac{-1}{2} \times 16 = -8$.

Q. Prove that $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies $A^2 - 4A + I = 0$ using it find A^{-1}.

$\because$ square matrix of order $= 2$

satisfies then the eqn.

$x^2 - (\text{tr}A)x + |A| = 0$

$x^2 - 4x + 1 = 0$

$A^2 \cancel{-4A}\ 4x + 1 = 0$.

$\boxed{A^2 - 4A + I = 0}$

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Q. Find matrix A satisifying the equation.

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}}_{B} A \underbrace{\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}}_{D}$$ ques

Sol^n

$$B^{-1}ABC = B^{-1}DC^{-1}$$

$$\Rightarrow A = B^{-1}C^{-1}D.$$

$$A = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}\left(\frac{1}{(-1)}\begin{bmatrix} \cancel{3} & -2 \\ -5 & \cancel{3} \end{bmatrix}\right)$$

$$= \begin{bmatrix} -7 & 9 \\ 12 & -14 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}$$

Solving a system of linear equations (Matrix Method)

Let eq^n be
$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Above equations can be written as matrix eq^n $A\,\cancel{x} = B$

Matrix $AX = B$

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Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

1) If $|A| \neq 0$ then system of eqn will be consistent and will have unique soln.

2) If $|A| = 0$ and $\text{adj}(A) \cdot B = 0$ then system of eqn of will be consistent and will have ∞ no of soln.

3) If $|A| = 0$ ~~adj~~ $\text{adj}(A) \cdot B \neq 0$ then eqns will be in consistent i.e. Having no soln.

Q. Solve the eqns using matrix method.

$$x + y + z = 6$$
$$y - z + z = 2$$
$$2x + y - z = 1$$

$$AX = B$$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$$X = A^{-1}B = \frac{1}{|A|} \text{adj}(A) \cdot B.$$

$$|A| = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} = 1(1-1) - 1(-1-2) + (1+2).$$
$$= 6.$$

$C_{11} = (1-1) = 0$, $C_{12} = -(-1-2) = 3$

$C_{13} = (1+2) = 3$, $C_{21} = -(-1-1) = 2$

$C_{22} = -1-2 = -3$, $C_{23} = -(1-2) = 1$

$C_{31} = (1+1) = 2$, $C_{32} = -(1-1) = 0$ $C_{33} = -1-1 = -2$

$$\begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix} = \text{Matrix of cofactors.}$$

$$\therefore X = \frac{1}{6} \begin{bmatrix} 0 & 0 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1$ $y = 2$ $z = 3$

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